1.1 Quadratic sequences

Generate the terms of a quadratic sequence

Key words
quadratic sequence
general term
variable
squared
term

A **quadratic sequence** is one with a **general term** that includes a **variable** that is **squared**.

For example: $T(n) = n^2 + 5$ is the general term of a quadratic sequence.
To find the **terms** of a quadratic sequence we substitute the term number into the expression.

Example 1

Find the first five terms of the sequence $T(n) = 3n^2 - 4$.

$T(1) = 3 \times 1^2 - 4 = 3 - 4 = ^{-}1$
$T(2) = 3 \times 2^2 - 4 = 12 - 4 = 8$
$T(3) = 3 \times 3^2 - 4 = 27 - 4 = 23$
$T(4) = 3 \times 4^2 - 4 = 48 - 4 = 44$
$T(5) = 3 \times 5^2 - 4 = 75 - 4 = 71$

Substitute $n = 1, 2, 3, 4$ and 5 into the general term. Don't forget to follow the order of operations.

Example 2

a) Write down the first five terms of the sequence $T(n) = n^2$.
b) Without substituting the term numbers in find the first five terms of the sequence $T(n) = 2n^2$.

a) $T(1) = 1^2 = 1$
$T(2) = 2^2 = 4$
$T(3) = 3^2 = 9$
$T(4) = 4^2 = 16$
$T(5) = 5^2 = 25$

The square numbers are $1^2, 2^2, 3^2, \dots$ and so on.

b) $T(1) = 2 \times 1 = 2$
$T(2) = 2 \times 4 = 8$
$T(3) = 2 \times 9 = 18$
$T(4) = 2 \times 16 = 32$
$T(5) = 2 \times 25 = 50$

If we know the first five terms of the sequence $T(n) = n^2$, to find the first five terms of the sequence $T(n) = 2n^2$ we must simply multiply all the terms of the sequence $T(n) = n^2$ by 2.

Exercise 1.1

1 Find the first five terms of each of the following sequences:

a) $T(n) = 2n^2 + 3$ **b)** $T(n) = n^2 - 10$ **c)** $T(n) = 3n^2 - 5$
d) $T(n) = 2n^2 + 10$ **e)** $T(n) = n^2 + 0.5$

2 Copy and complete the table:

Sequence \ Term Number	1	2	3	4	5
a) $T(n) = n^2$					
b) $T(n) = n^2 + 1$					
c) $T(n) = n^2 - 3$					
d) $T(n) = 3n^2$					
e) $T(n) = 0.5n^2$					

You should be able to fill in the table with very little calculation. See Example 2.

3 A quadratic sequence can be generated on a spreadsheet. For example, to generate the sequence $T(n) = 3n^2 + 5$ you would enter the following into a spreadsheet:

	A	B
1	1	= 3*A1^2+5
2	2	= 3*A2^2+5
3	3	= 3*A3^2+5
4	4	= 3*A4^2+5

a) Find the first ten terms of the sequence $T(n) = 3n^2 + 5$.

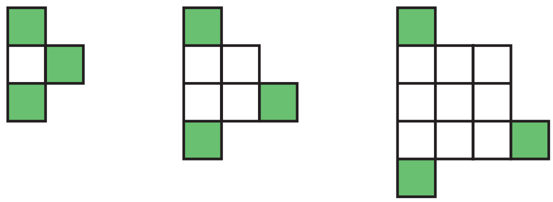
b) Use a spreadsheet to find the first ten terms of each of the sequences in **Q1**.

4 The general term of a sequence is: $T(n) = \dfrac{n}{n^2 + 1}$

The first term of the sequence is: $T(1) = \dfrac{1}{1^2 + 1} = \dfrac{1}{2}$

Find the first five terms of the sequence. Leave your answers as fractions.

5 **a)** Draw the next diagram in this sequence.



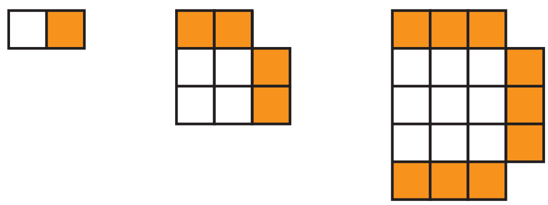
b) Copy and complete this table:

Pattern number	1	2	3	4
Number of white squares				
Number of green squares				
Total number of squares				

c) What is the general term for the number of white squares?

d) What is the general term for the total number of squares?

6 Find the general term for the number of squares in the terms of this sequence. Usie a similar method to the one you used for **Q5**.



7 **a)** Write down the first six terms in the sequence $T(n) = n^2$.

b) Find the differences between consecutive terms.

c) Describe any patterns you notice.

d) Repeat for the following sequences:

i) $T(n) = n^2 + 1$ **ii)** $T(n) = n^2 + 10$ **iii)** $T(n) = n^2 - 3$ **iv)** $T(n) = n^2 - 1$

e) Describe any patterns in the differences between consecutive terms that apply to all sequences of the form $T(n) = n^2 \pm a$, where a is a number.

8 **a)** Write down the first six terms of each of the following sequences:

i) $T(n) = n^2$ **ii)** $T(n) = 2n^2$ **iii)** $T(n) = 3n^2$

b) Look at the differences between consecutive terms and describe any patterns you notice.

c) Predict what the difference between consecutive terms will be for the sequence $T(n) = 4n^2$.

d) Check your prediction.

1.2 The general term

Find the general term of a quadratic sequence

Key words
general term
consecutive terms
quadratic

To find the **general term** of a linear sequence, look at the difference between **consecutive terms** and compare the sequence to the multiples of that difference.

We noticed in Lesson 1.1 that for all **quadratic** sequences of the form $T(n) = n^2 + an + b$ where a and b are numbers, the second row of differences is 2.

For example the first five terms of the sequence $T(n) = n^2 + 2n - 3$ are:

$T(\mathbf{1}) = \mathbf{1}^2 + 2 \times \mathbf{1} - 3 = 1 + 2 - 3 = 0$
$T(\mathbf{2}) = \mathbf{2}^2 + 2 \times \mathbf{2} - 3 = 4 + 4 - 3 = 5$
$T(\mathbf{3}) = \mathbf{3}^2 + 2 \times \mathbf{3} - 3 = 9 + 6 - 3 = 12$
$T(\mathbf{4}) = \mathbf{4}^2 + 2 \times \mathbf{4} - 3 = 16 + 8 - 3 = 21$
$T(\mathbf{5}) = \mathbf{5}^2 + 2 \times \mathbf{5} - 3 = 25 + 10 - 3 = 32$

Look at the differences between consecutive terms.
The second row of differences is constant: it is always 2.

0 5 12 21 32
+5 +7 +9 +11
+2 +2 +2

If the second row of differences of a sequence is constant then we can find the general term of that sequence by comparing it to the sequence of square numbers.

For example, to get the rule above, subtract the square numbers and find the linear rule of the remaining sequence.

Example

Find the general term of the sequence 4, 7, 12, 19, 28, 39,

4 7 12 19 28 39
+3 +5 +7 +9 +11
+2 +2 +2 +2

Look at the differences: if they are not the same, look at the second row of differences. Here, the differences are constant, so the sequence is quadratic.

The square numbers are:	1, 4, 9, 16, 25, 36, …
The sequence is:	4, 7, 12, 19, 28, 39, …
The general term is:	$T(n) = n^2 + 3$

Compare the sequence to the sequence of square numbers $T(n) = n^2$.

Each term in the sequence is three more than a term in the sequence $T(n) = n^2$.

Exercise 1.2

1 Find the general term of the linear sequences below:

a) $^-10$, $^-12$, $^-14$, $^-16$, …
b) $^-25$, $^-24$, $^-23$, $^-22$,…
c) 3.2, 3.7, 4.2, 4.7,…
d) 0.9, 0.8, 0.7, 0.6, …

2 The first ten terms of a sequence are 6, 9, 14, 21, 30, 41,…

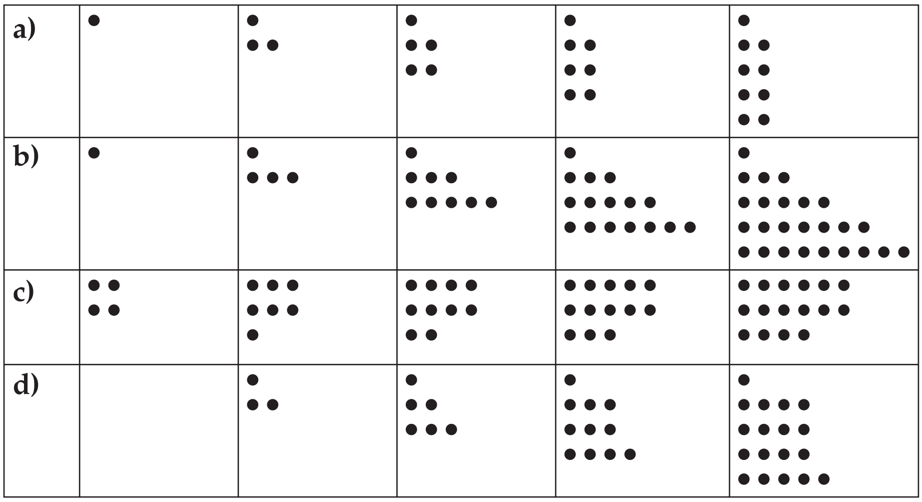
a) What are the differences between consecutive terms?
b) What are the differences between the differences (the second row of differences)?
c) What does your answer to part **b)** tell you?
d) Find the general term of the sequence.

3 Find the general term of each of the sequences below:

Look at the Example.

a) 2, 5, 10, 17, 26, 37, …
b) 0, 3, 8, 15, 24, 35, …
c) 13, 16, 21, 28, 37, 48, …
d) $^-9$, $^-6$, $^-1$, 6, 15, 26, …
e) 1.5, 4.5, 9.5, 16.5, 25.5, 36.5, …
f) $\frac{3}{4}$, $3\frac{3}{4}$, $8\frac{3}{4}$, $15\frac{3}{4}$, $24\frac{3}{4}$, $35\frac{3}{4}$, …

4 Find the general term for the number of dots in each of the following sequences:



First decide whether they are linear or quadratic sequences.

5 **a)** Find the second row of differences between consecutive terms for the following sequences:

i) $T(n) = 2n^2$ **ii)** $T(n) = 2n^2 + 3$ **iii)** $T(n) = 2n^2 - 5$
iv) $T(n) = 2n^2 + n$ **v)** $T(n) = 2n^2 + 3n - 1$

b) What do you notice about the second row of differences for all these sequences?

6 Repeat **Q5** but replace the number 2 with the number 3 throughout.

7 Copy and complete the table below.

You might find it helpful to choose some particular sequences to test out.

Sequence	Second difference
a) $T(n) = n^2 + an + b$	2
b) $T(n) = 2n^2 + an + b$	
c) $T(n) = 3n^2 + an + b$	
d) $T(n) = 4n^2 + an + b$	
e) $T(n) = 5n^2 + an + b$	

f) Look at the completed table. What is the relationship between the second difference and the number in front of the n^2?

Investigation

8 Choose any sequence of the form $T(n) = an^2 + bn + c$ where a, b and c are numbers.

Is the second row of differences always constant (the same number).

An example is $T(n) = 3n^2 + 8n + 7$.

1.3 Special sequences

Find the general term of some more unusual sequences by looking at number and geometric patterns

Key words
linear
quadratic
square number

Remember: one way to find the general term of a sequence is to look at the differences.

1. If the first differences are constant (the same number) then the sequence is **linear** and to find the general term we compare the sequence to the multiples of the difference.
2. If the second differences are constant then the sequence is **quadratic** and we compare the sequence to the sequence of **square numbers**.

Sometimes it is difficult to find the general term by comparing the sequence to multiples or square numbers as they don't fall into either category. For example, 1 1 2 3 5 8 …

When this is the case, we can look at the structure of the sequence.

Example 1

Find the general term and the 12th term of the sequence: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Numerators: 1, 2, 3, 4, …

Denominators: 2, 3, 4, 5, ….

The general term is: $T(n) = \frac{n}{n+1}$

The 12th term is: $T(12) = \frac{12}{12+1} = \frac{12}{13}$

When finding the general term of a sequence of fractions consider the numerators and denominators separately.

The general term of the numerator is $T(n) = n$. The denominator is always one more than the numerator so the general term of the denominator is $T(n) = n + 1$.

Example 2

An alien from Mars takes two weeks to reach adult size. It is born with 2 legs and each day during its growth the number of legs it has doubles. How many legs does an adult alien have?

Day	1	2	3	4	5
Number of legs	2	4	8	16	32
	2	2×2	$2 \times 2 \times 2$	$2 \times 2 \times 2 \times 2$	$2 \times 2 \times 2 \times 2 \times 2$

First draw up a table to show what happens in the first few days of growth. Consider how the sequence has developed.

$T(n) = 2^n$

An adult alien will have $T(14) = 2^{14} = 16384$ legs

To find each term of the sequence you multiply two by itself n times.

Exercise 1.3

Look at Example 1.

1 a) Find the general term of each of the following sequences:

i) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$ ii) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \ldots$ iii) $\frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{8}{5}, \ldots$ iv) $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots$.

b) Use your answer to part **a)** to find the 15th term of each of the sequences.

2 The Fibonacci sequence has first term 1 and second term 1. The next term is found by adding the previous two terms i.e. 1, 1, 2, 3, …

a) Find the first ten terms of the Fibonacci sequence.

Other 'Fibonacci-like' sequences can be found by choosing the first term and second term and then following the same rule.

b) Find the first six terms of each of the following 'Fibonacci-like' sequences if the first and second terms are:

i) 2, 2 **ii)** 1, 2 **iii)** 3, 5 **iv)** 10, 100 **v)** a, b

c) A 'Fibonacci-like' sequence has second term 20 and fifth term 80. Find the first term.

Your answer to part **b v)** might help.

3 The first five terms of a sequence are:

6, 12, 20, 30, 42, 56

a) Look at the difference between consecutive terms, what does this tell you about the general term of the sequence?

b) Copy and complete the table illustrating the sequence by drawing rectangular patterns of dots:

Term	1	2	3	4	5	6
Sequence	●● ●● ●●	●●● ●●● ●●● ●●●	●●●● ●●●● ●●●● ●●●●			
Height of triangle	3	4	5			
Width of triangle	2	3	4			
Number of dots	3×2	4×3				

c) How many dots will there be in the nth term of the sequence?

d) What is the 12th term of the sequence?

4 A frog sits in the middle of a lily pad of radius 1 m. It takes a series of jumps towards the edge of the pad. On the first jump it jumps halfway towards the edge of the table, on the second it jumps half the remaining distance and so on.

How far does the frog jump in its

a) **i)** first jump **ii)** second jump **iii)** third jump **iv)** fourth jump **v)** nth jump?

b) How many jumps will it take to reach the water?

5 A yoghurt left out of the fridge begins to grow bacteria.

On day 1 it has one cell. Each day the number of cells it has triples.

See Example 1.

a) Work out how many cells of bacteria it has on day 4.

b) Find the nth term of the sequence.

c) How many cells will be present on day 100?

Investigations

6 A regular hexagon has diagonals drawn such that each vertex is joined to each of the other vertices.

a) How many diagonals are drawn?

b) Investigate for other shapes.

You could investigate triangles, squares, pentagons, … and so on.

c) Find the general rule connecting the number of sides in a shape to the number of diagonals drawn.

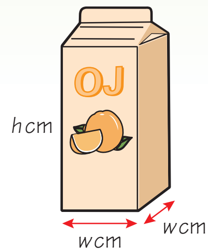
1.4 Formulae

- Derive a formula
- Change the subject of a formula

Key words
formula
variable
subject
derive

A **formula** is a general rule that expresses a relationship between independent and dependent **variables**.
The **subject** of a formula is the unknown which is alone on one side of the equation.
For example: in the formula $v = ut + \frac{1}{2}at^2$ the subject is v.
We can **derive** formulae by considering the general case of a given situation.

Example



a) Derive a formula for finding the volume (V) of orange juice that this carton can hold (in litres).

b) Rearrange the formula to make h the subject.

c) Find the height for a carton given that $w = 10$ and the carton must hold 3 litres of orange juice.

a) Volume of carton $= h \times w \times w$ — First find the volume of the carton in cm^3.

$V = \frac{h \times w \times w}{1000}$ — $1\ cm^3 = 1$ ml. Since 1000 ml $= 1\ \ell$ to find the volume of the carton in litres we must divide the volume by 1000.

$V = \frac{hw^2}{1000}$ — We can write $w \times w$ as w^2.

b) $V = \frac{hw^2}{1000}$ — To make h the subject of the formula we must end up with the h alone on one side of the equation.

$1000V = hw^2$ — Multiply both sides by 1000.

$\frac{1000V}{w^2} = h$ — Divide both sides by w^2.

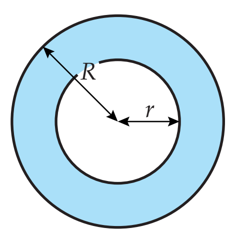
c) $\frac{1000 \times 3}{10^2} = h$ — Substitute $V = 3$ and $w = 10$ into the formula.

$\frac{3000}{100} = h$ — h is now the subject of the equation.

$30\ cm = h$

Exercise 1.4

Give your answers to 1 decimal place.



Area of a circle $= \pi \times \text{radius}^2$

1 This shape is called an annulus. It is made up of a circle of radius R with a circular hole removed from it, of radius r.

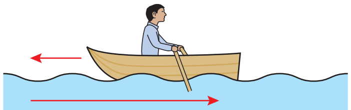
a) Find the area of the annulus when:

i) $r = 3$ cm, $R = 10$ cm

ii) $r = 9$ m, $R = 12$ m

b) Derive a formula for finding the area of an annulus, A.

2 Mr Murphy wishes to row up a river. He can row at a speed of 12 km/hr. The rate at which the river flows varies according to the weather.



Remember that Mr Murphy is rowing against the flow of the river.

a) At what speed does Mr Murphy travel when the river flows at a rate of:

i) 3 km/hr **ii)** 5 km/hr **iii)** 9 km/hr?

b) Derive a formula for calculating the speed of the boat, S, when the speed at which the river flows is v.

c) Make v the subject of the equation and use this to calculate the speed of the river when Mr Murphy travels at a speed of:

i) 10 km/hr **ii)** 8 km/hr **iii)** 1 km/hr

3 A car can accelerate at a rate of 10 metres per second squared.

Acceleration can be measured in metres per second squared.

a) If the car starts from stationary what is its speed after accelerating for:

i) 5 seconds **ii)** 2 seconds **iii)** 10 seconds?

b) Write down a formula for the speed at which the car is travelling, V, after t seconds.

c) Make t the subject of the formula and use this to calculate how long the car has been accelerating for if its speed is:

i) 43 metres per second **ii)** 25 metres per second **iii)** 70 metres per second?

4 **a)** Derive a formula for calculating the perimeter of a semicircle, P, when the radius is r.

It might help to draw a diagram.

b) Use your formula to find the perimeter of a semicircle when:

i) radius = 2 cm **ii)** radius = 10 cm **iii)** diameter = 5 cm

c) Factorise your formula.

We factorise an algebraic expression by finding a common factor.

d) Rearrange the equation to make r the subject and hence find the radius of the the semicircle when the perimeter is:

i) 100 cm **ii)** 55 cm **iii)** 18 cm

Investigation

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

5 Crosses are made on a 5×5 grid in the following way.

The value of a cross is shown by $C(n)$.

$C(7)$ is calculated by adding all the values in the cross:
$C(7) = 2 + 6 + 7 + 8 + 12 = 35$

a) Find the value of: **i)** $C(8)$ **ii)** $C(9)$ **iii)** $C(12)$

b) If the middle of the cross has value n, what is the value of the square directly above it?

c) Use this idea to find a formula for $C(n)$.

d) Investigate how the size of the grid alters the value of $C(n)$. For example, find $C(n)$ for a 6×6 grid, a 5×6 grid and so on.

1.5 The inverse of a linear function

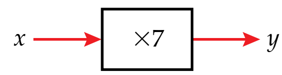
- Find the inverse of a linear function
- Plot the graph of a function and its inverse

Key words
function
inverse

A **function** is a way of expressing a relationship between two sets of values.

A function can be expressed in three different ways:

- as a function machine:
- in words: '*y* is always seven times *x*'
- algebraically: $y = 7x$



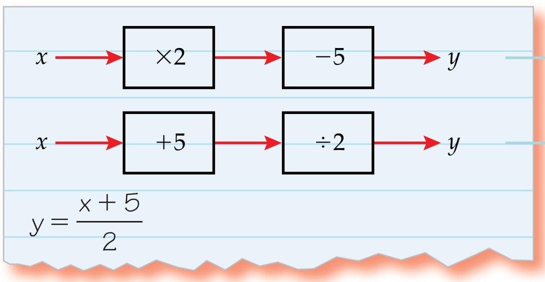
The **inverse** of a function 'undoes' what the function has done.

To find the **inverse** we carry out **inverse** operations in reverse order.

The **inverse** of the function above will be $y = \frac{x}{7}$ since dividing by 7 is the **inverse** of multiplying by 7.

Example

Find the inverse of the function: $y = 2x - 5$



Drawing a function machine shows that *x* is first multiplied by 2 and then 5 is subtracted.

To find the inverse function we must carry out the inverse operations in reverse order.

Exercise 1.5

1 Find the inverse of each of the following functions:

a) $y = 2x$ **b)** $y = x + 4$ **c)** $y = x - 5$

d) $y = \frac{x}{9}$ **e)** $y = -3x$ **f)** $y = 7 + x$

2 **a)** Draw a function machine to illustrate the function $y = \frac{x}{5} + 1$.

b) Draw a function machine to show the inverse of the function $y = \frac{x}{5} + 1$.

c) Write down the inverse of the function $y = \frac{x}{5} + 1$.

3 Find the inverse of each of the following functions:

It may help to draw function machines.

a) $y = 2x + 1$ **b)** $y = \frac{x + 5}{4}$ **c)** $y = 3(x - 10)$

d) $y = \frac{x}{3} - 5$ **e)** $y = 7x - 12$ **f)** $y = 2(x + 5)$

g) $y = \frac{x - 9}{15}$ **h)** $y = \frac{x}{12} + 4$ **i)** $y = 22x - 1.5$

4 To convert from kilometres to miles we use the following function:

$y = 1.6x$, where x is kilometres and y is miles.

a) Convert the following distances from miles to kilometres:

i) 10 miles **ii)** 8 miles **iii)** 5 miles

b) Find the inverse of the function.

c) Use your answer to part **c)** to convert the following distances into miles:

i) 20 km **ii)** 8 km **iii)** 60 km

5 To convert an area from m^2 to cm^2 we use the following function:

$y = 10000x$

a) Convert the following areas from m^2 to cm^2:

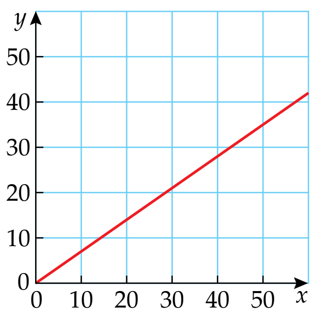
i) 0.5 m^2 **ii)** 0.0021 m^2 **ii)** 0.007 m^2

b) Find the inverse of the function.

c) Use your answer to part **c)** to convert the following areas from cm^2 to m^2.

i) 30 000 cm^2 **ii)** 69 000 cm^2 **iii)** 89 700 cm^2

6 The following graph is used to convert from pounds, on the x-axis to euros, on the y-axis:



Look at the gradient and the y-intercept.

a) What is the equation of the graph?

b) Find the inverse of the function, and use this to calculate the value in pounds of:

i) € 70 **ii)** € 250 **iii)** € 380

7 You may use a graph sketching package for this question.

a) Draw a pair of axes where $0 \leqslant x \leqslant 10$ and $0 \leqslant y \leqslant 10$.

b) On the axes, draw the graph of $y = 2x$ and its inverse $y = \frac{1}{2}x$.

c) Draw the graph of $y = x$.

d) What do you notice about the relationship between the graph $y = 2x$ and its inverse?

Look for any lines of symmetry.

e) Experiment with other graphs and their inverse. Does your answer to part **d)** hold?

8 A function is called 'self inverse' if both the function and its inverse are identical.

a) Find a self inverse function.

b) Is there more than one self inverse function? Explain your answer.

1.6 Graphs

Plot the graphs of linear functions

Key words
equation
plot
axes

You have seen graphs of the form $y = mx + c$, where m = gradient and c = y-intercept.

The **equation** of a straight-line graph can be written in other ways.

For example, we can write the equation of a straight-line graph in the form:
$y - 3x + 4 = 0$

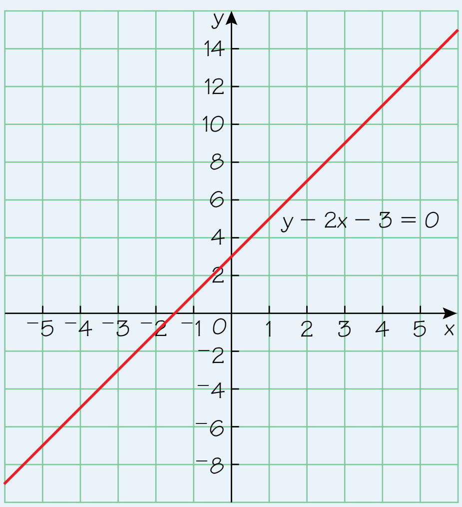
When we **plot** the graph of these equations we follow exactly the same method as when we plot graphs of the form $y = mx + c$.

Example

Draw the graph of $y - 2x - 3 = 0$, choosing x-values between $^-5$ and 5.

x	$^-5$	0	5
	$y - 2x - 3 = 0$ $y - 2 \times {}^-5 - 3 = 0$ $y + 10 - 3 = 0$ $y + 7 = 0$ $y = {}^-7$	$y - 2x - 3 = 0$ $y - 2 \times 0 - 3 = 0$ $y + 0 - 3 = 0$ $y - 3 = 0$ $y = 3$	$y - 2x - 3 = 0$ $y - 2 \times 5 - 3 = 0$ $y - 10 - 3 = 0$ $y - 13 = 0$ $y = 13$
y	$^-7$	3	13

First draw up a table and choose some x-values. Calculate the y-value by substituting the x-value into the equation and solving to find y.



Draw a suitable pair of **axes** and plot the points. Join them with a straight line.

Exercise 1.6

Choose x-values between $^-5$ and 5.

1 On graph paper plot the following graphs:

a) $y - 4x = 0$ **b)** $y - x = 10$

c) $2y + 2x = 11$ **d)** $\frac{y}{2} + x = 7$

2 **a)** Copy and complete the table of values for the graph $\frac{x}{3} + \frac{y}{2} = 1$:

x	$^-6$	0		
y			$^-2$	$^-4$

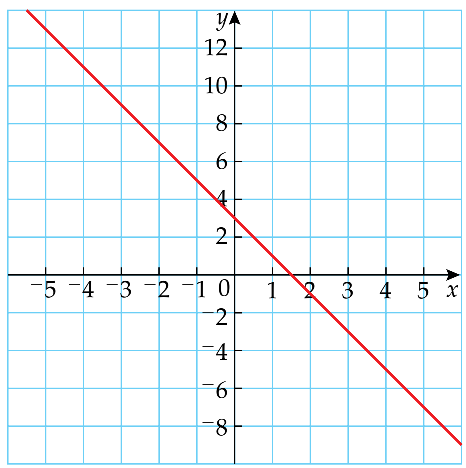
b) Hence draw the graph of $\frac{x}{3} + \frac{y}{2} = 1$.

3 Decide which of the following points lie on the graph $y + x + 1 = 0$.

a) (1, 1) **b)** (3, -4) **c)** ($^-1$, $^-2$)

d) ($^-2$, $^-4$) **e)** (7, $^-8$) **f)** ($^-10$, 11)

4 This is the graph of $2y + 4x - 6 = 0$.



a) Where does the graph cross the y-axis?

b) Calculate the gradient of the graph.

c) Use your answers to parts **a)** and **b)** to write down the equation of the graph in the form $y = mx + c$.

d) Rearrange the equation of the graph to make y the subject.

f) What do you notice about your answers to parts **c)** and **d)**?

5 **a)** Rearrange each of the following equations to make y the subject:

i) $2y - 8x = 10$ **ii)** $y + 4x - 7 = 0$ **iii)** $3y = {}^-6x + 12$

b) Use your answer to part **a)** to find the gradient and the y-intercept of each of the graphs.

6 You may use a graph sketching package for this question:

a) Make y the subject of the equation $5y = 12x - 3$.

b) Find the inverse of this function.

c) Check your answers to parts **a)** and **b)** by plotting the graphs.

Investigation

7 **a)** Find the gradient of the graph $2y + 3x = 8$.

b) Investigate how to find the gradient of other graphs of the form $ay + bx = c$ where a, b and c are integers. Can you find a general rule?